

The Aharonov-Bohm effect in spectral asymptotics of the magnetic Schrödinger operator

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Abstract: We show that in the absence of a magnetic field the spectrum of the magnetic Schrödinger operator in an annulus depends on the cosine of the flux associated with the magnetic potential. This result follows from an analysis of a singularity in the “wave trace” for this Schrödinger operator, and hence shows that even in the absence of a magnetic field the magnetic potential can change the asymptotics of the Schrödinger spectrum, i.e. the Aharonov-Bohm effect takes place. We also study the Aharonov-Bohm effect for the magnetic Schrödinger operator on a torus.

§1. Introduction.

Let Ω be the exterior of a bounded region in \mathbb{R}^2 with smooth boundary, and let

$$H_{A,V} = \frac{1}{2}(-i\partial_{x_1} + A_1(x))^2 + \frac{1}{2}(-i\partial_{x_2} + A_2(x))^2 - V(x) \quad (1)$$

This is the Schrödinger operator for a particle of mass 1 and charge -1 moving in Ω under the influence of the magnetic potential $A = (A_1, A_2)$ and the electric potential V . We assume that

$$\partial_{x_2} A_1 - \partial_{x_1} A_2 = 0 \text{ in } \Omega, \quad (2)$$

i.e., the magnetic field vanishes in Ω . Given a simple, closed curve γ in Ω encircling the complement of Ω , we define the magnetic flux by

$$\alpha_\gamma = \int_\gamma A(x) \cdot dx.$$

In view of (2) α_γ only depends on the orientation of γ .

In the seminal paper [AB] Aharonov and Bohm showed that if $\alpha_\gamma \neq 0 \bmod 2\pi$, then one can detect the cosine of the magnetic flux in the scattering of particles in this quantum system, i.e. the magnetic potential has a physical impact even when the magnetic field is zero in Ω . This effect is called the Aharonov-Bohm effect. They computed the scattering cross-section explicitly for $\Omega = \mathbb{R}^2 \setminus \{0\}$, when $A(x) = (-x_2/|x|^2, x_1/|x|^2)$ and $V(x) = 0$. They also proposed an experiment to demonstrate this effect. However, the first generally accepted experimental verification of the Aharonov-Bohm (AB) effect was done many years later by Tonomura et al. [T]. For further mathematical work on the AB effect see [N], [W], [RY], [E2], [E3], [EIO].

In [H] Helffer showed that $A(x)$ can influence the spectrum of $H_{A,V}$ when the magnetic field is zero in Ω . In the semi-classical setting with $V(x) \rightarrow \infty$, as $|x| \rightarrow \infty$, and $\Omega = \{|x| > 1\}$ he showed that the lowest Dirichlet eigenvalue depended on the cosine of the magnetic flux. Earlier related results on magnetic Schrödinger operators are due to Lavine and O’Carroll ([L-C]).

In this paper we study the Schrödinger operator in the domain $\Omega_R = \Omega \cap \{|x| < R\}$ with Dirichlet boundary conditions on $|x| = R$ and $\partial\Omega$. We compute the singularity at $t = 3R\sqrt{3}$ of the distribution trace of the fundamental solution of the initial-boundary value problem

$$u_{tt} + H_{A,V}u = 0, \quad u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad u(x, t) = 0 \text{ when } x \in \partial\Omega_R. \quad (3)$$

This distribution trace is known as the “wave trace” for this problem, and it is given by

$$\sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j}),$$

where $\{\lambda_j\}_{j=1}^{\infty}$ are the Dirichlet eigenvalues of $H_{A,V}$ in Ω_R . Hence its singularities are determined by the behavior of the λ_j as $j \rightarrow \infty$. These singularities are well-known to appear only at the lengths of periodic broken ray paths in Ω_R . The singularity at $t = 3R\sqrt{3}$ comes from equilateral triangles in Ω_R with vertices on $|x| = R$. To compute this singularity we need to know that $3R\sqrt{3}$ is isolated in the set of lengths of broken periodic rays. To ensure that we assume that the complement of Ω , Ω^c , is strictly convex and contained in $\{|x| < 1\}$ and $R \geq 8$ (see Remark I.1), but any assumption that makes the length of the inscribed equilateral triangles isolated in the lengths of periodic reflected ray paths will suffice. The geometry that we have chosen makes the singularity unchanged when one changes the sign of α_γ . Hence we cannot recover more than the cosine of α_γ from it (see Remark I.2).

A definitive computation of leading singularities in wave traces was given by Duistermaat and Guillemin in [DG] for manifolds without boundary. For manifolds with boundary the analogous computation has not been done in that generality. To carry it out in here we have taken this opportunity to present a different method of computation that replaces Fourier integral operators with superpositions of Gaussian beams (cf. [CRR] and Chapter 5 of [CR]). The result of the computation is that the leading singularity at $t = L = 3R\sqrt{3}$ is the real part of the distribution

$$\pm \pi^3 R (2R)^{1/2} 3^{1/4} \cos\left(\int_{\gamma} A(x) \cdot dx\right) \frac{1}{(t - L - i0)^{3/2}},$$

where the initial \pm does not depend on A .

In the final section of this paper we consider $H_{A,V}$ on (flat) 2-torus and obtain essentially the same result: under a nondegeneracy assumption on the torus the singularities in the wave trace at times equal to the lengths of curves in a homology basis determine the cosines of magnetic fluxes around those curves (see Theorem 5.1).

Remark I.1: The only fact from geometry needed here is (and we only need it for circles): rays and their reflections inside an ellipse are tangent to the same ellipse confocal with the boundary ellipse. So rays in $|x| \leq R$ tangent to a circle $|x| = r > 1$ will never enter $|x| < 1$ after reflection in $|x| = R$, while rays that enter $|x| < 1$ will always re-enter $|x| < 1$ after reflection in $|x| = R$. Since the boundary curve C is convex, rays entering $|x| < 1$ will leave $|x| < 1$ after at most one reflection. This gives the following bounds on the length L of periodic ray paths that hit C . For rays that close after entering $|x| < 1$ k times

$$2kR - 2k < L < 2kR + 2k.$$

So periodic rays that enter $|x| < 1$ more than three times have lengths are greater than $8R - 8$, and the equilateral triangles are the (isolated) shortest periodic rays that never enter $|x| < 1$ (assuming $R > 2$). So we need $4R + 4 < 3R\sqrt{3} < 6R - 6$. That happens as soon as $R \geq 8$ (picking the first whole number that works).

Remark I.2: If $\Omega = \{|x| > 1\}$ and $V \equiv 0$, the mapping $u(x) \rightarrow u(-x)$ sends eigenfunctions of $H_{A,0}$ to eigenfunctions of $H_{-A,0}$ bijectively. Thus the wave traces of these operators must be identical. The leading singularity in the wave trace at $t = 3\sqrt{3}R$ does not depend on the boundary of Ω or $V(x)$, hence it will be unchanged when A is replaced by $-A$ in these cases, too. Therefore, one cannot distinguish α_γ and $-\alpha_\gamma$ using the leading singularity. The same ambiguity arises in the results in [AB] and [H].

§2. Singularities of the Wave Trace.

Let $E(x, y, t)$ denote the fundamental solution for the initial-boundary value problem (3). The wave front set of the distribution kernel of E is contained in the canonical relation for the bicharacteristic flow (see Melrose-Sjöstrand, [MS I, II]). For this problem this canonical relation is defined as follows: Let $\nu(x)$ denote the outer unit normal to $\partial\Omega_R$ at x . Given (y_0, η_0) with $y_0 \in \Omega_R$ and $|\eta_0| = 1$, define $(x(s, y, \eta), \xi(s, y, \eta)) = (y + s\eta, \eta)$ until, at $s = s_1$, $y_1 = x(s_1, \eta_0, y_0) \in \partial\Omega_R$. Then, if $\eta \cdot \nu(y_1) \neq 0$, continue $(x(s, y_0, \eta_0), \xi(s, y_0, \eta_0))$ for $s > s_1$ as $(y_1 + s\eta_1, \eta_1)$, where $\eta_1 = \eta_0 - 2(\nu(y_1) \cdot \eta_0)\nu(y_1)$. Continue the bicharacteristic this way, reflecting when $x(s, y_0, \eta_0)$ hits $\partial\Omega_R$, as long as $x(s, y_0, \eta_0)$ does not intersect $\partial\Omega_R$ tangentially. At points of tangential intersection one has to distinguish grazing and gliding points. However, since we assume that the boundary of Ω^c is strictly convex, points of tangential intersection with $\partial\Omega$ are grazing points and bicharacteristics continue unaffected by these intersections. When y_0 is in the interior of Ω_R , a bicharacteristic with initial data (y_0, η_0) will never intersect $|x| = R$ tangentially. Hence, the wave front set of the kernel of $E(\cdot, \cdot, t)$ is the union over $y_0 \in \Omega_R$ and $\eta_0 \in \mathbb{S}^1$ of the points

$$(x(t, y_0, \eta_0), \xi(t, y_0, \eta_0), y_0, -\eta_0),$$

where $(x(t, y_0, \eta_0), \xi(t, y_0, \eta_0))$ are the reflected bicharacteristics described above. Strictly speaking, the wave front set is the closure of that set and includes a “boundary wave front set” over $|x| = R$ (see [MS] for details).

Since $E(x, y, t)$ is a distribution in t depending smoothly on $(x, y) \in \Omega_R \times \Omega_R$, $\int_{\Omega_R} E(x, x, t) dx$ is well-defined, and we have the following relation

$$T =_{def} \sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j}) = \int_{\Omega_R} E(x, x, t) dx.$$

The singular support of T is contained in the set of t such that $(y_0, \eta_0, y_0, -\eta_0) \in WF(E(x, y, t))$ for some $y_0 \in \overline{\Omega_R}$, [GM]. The choice of Ω and R here implies that, for t in a sufficiently small neighborhood of $3R\sqrt{3}$, $(y_0, \eta_0, y_0, -\eta_0) \in WF(E(x, y, t))$ only if the ray $x(s, y_0, \eta_0)$ traces an inscribed equilateral triangle.

To compute the singularities in the wave trace we need a parametrix for the initial-boundary value problem (3). Since this parametrix will differ from $E(x, y, t)$ by an integral operator with a smooth kernel, we can use it to compute singularities. Since we are only interested in singularities arising from inscribed equilateral triangles, we only need a parametrix which captures the singularities of $\int E(x, y, t) f(y) dy$ when $WF(f) \subset \{y, \eta) : y \in \Omega_R, |y \cdot \eta^\perp| = R/2\}$, where $(\eta_1, \eta_2)^\perp = (\eta_2, -\eta_1)$. These singularities hit $\partial\Omega_R$ nontangentially, and hence this parametrix construction can be done with reflection at the boundary. This observation applies equally well to constructions with Fourier integral operators and the Gaussian beam superpositions used here.

§3. The Gaussian beam construction. Here we will outline the construction of a parametrix for (3), for initial data with wave fronts projecting onto the inscribed equilateral triangles. We will continue to let η have length one. The Gaussian beam method allows one to do the following (see [R] for more details):

i) For any ray, $(x(t), t) = (z + t\eta, t)$, in space-time, one can construct a function $\phi(x, t; z, \eta)$ satisfying:

(a) For any given integer N , $(\phi_t)^2 - |\phi_x|^2$ vanishes to order N on $(x(t), t)$ and $\text{Im}\{\phi_{xx}\}$ is positive definite on $(x(t), t)$.

(b) $\phi(x, 0; z, \eta) = x \cdot \eta + \frac{i}{2}|x - z|^2$ on $|x - z| < \delta$, and $\phi_t(x, 0; z, \eta) = -1$.

Moreover, if Γ is a curve with unit normal ν at $x(t_0)$ and η is not tangent to Γ , then one can construct $\phi^r = \phi$ on Γ , satisfying (a) for the reflected ray $(x(t_0) + (t - t_0)\eta^r, t)$, where $\eta^r = \omega - 2(\nu \cdot \eta)\nu$.

ii) Once ϕ has been constructed, for any given integer N , one can solve the transport equations

$$2\phi_t(a_0)_t - 2\phi_x \cdot (a_0)_x + (2iA(x) \cdot \phi_x + \phi_{tt} - \Delta\phi)a_0 = 0, \quad (4)$$

$$2\phi_t(a_j)_t - 2\phi_x \cdot (a_j)_x + 2iA(x) \cdot \phi_x + \phi_{tt} - \Delta\phi a_j = -(\partial_t^2 - (\partial_x + iA(x))^2)a_{j-1}, \quad j > 0$$

to order N on $(x(t), t)$, and impose the initial conditions $a_0(0, x; z, \eta) = 1$ and $a_j((0, x; z, \eta) = 0$ for $j > 0$ on $|x - z| < \delta$.

For the singularity computation we need to know the leading amplitude a_0 on the ray beginning at z in direction η .

We define $a(x, t; z, \eta, r)$ to be the formal sum

$$a(x, t; z, \eta, r) = \sum_{j \geq 0} a_j(x, t; z, \eta) r^{-j}. \quad (5)$$

As before one can reflect in a plane curve Γ which is transverse to the ray, and we impose $a^r = -a$ on Γ to satisfy Dirichlet boundary conditions.

Using the preceding constructions we can construct the operator

$$[V(t)f](x) = \frac{1}{2}([V_+(t)f](x) + [V_-(t)f](x)),$$

where

$$[V_{\pm}(t)f](x) = \sum_{k \geq 0} \frac{1}{(2\pi)^3} \int_{\mathbb{R}_+ \times S^1 \times \{|z| < R + \delta\}} e^{ir\phi^k(x, \pm t; z, \eta)} a^k(x, \pm t; z, \eta, r) \hat{f}(r\eta) r^2 dr d\eta dz. \quad (6)$$

Here, ϕ^0 is the phase function with $\phi^0(x, 0; z, \eta) = x \cdot \eta + \frac{i}{2}|x - z|^2$, and for $k > 0$,

$$e^{ir\phi^k(x, t; z, \eta)} a^k(x, t; z, \eta, r)$$

is the (Dirichlet) reflection of $e^{ir\phi^{k-1}(x, t; z, \eta)} a^{k-1}(x, t; z, \eta, r)$ in the circle $|x| = R$. Since Gaussian beams can be constructed to for any finite ray segment, we can assume that each term in (6) is defined on $\{|x| \leq 2R\}$ when necessary. Note that in this notation the variables (z, η) in ϕ^k remain the initial data **at $t=0$** for the ray where $\text{Im}\{\phi^k\} = 0$. Note also that the integration in r in (6) is in the sense of distributions.

For the parametrix construction we need $V(0)f = f + Kf$ where K is an operator with a smooth kernel. From (6) we have

$$[V(0)f](x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}_+ \times S^1 \times \{|z| < 2R\}} e^{irx \cdot \eta - r|x-y|^2/2} \hat{f}(r\eta) r^2 dr d\eta dz.$$

Since

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}_+ \times S^1 \times \mathbb{R}_z^2} e^{irx \cdot \eta - r|x-y|^2/2} \hat{f}(r\eta) r^2 dr d\eta dz = f(x)$$

and f is supported in $\{|x| < R\}$, it follows that omitting the contribution from $\{|z| > R + \delta\}$ in (6) only adds an operator with a smooth kernel.

To compute singularities of the wave trace we need to make the kernels of the operators $V_{\pm}(t)$ explicit. The distribution kernels of these operators are sums of terms of the form

$$S(t) = \int_{\mathbb{R}_+ \times S^1 \times \mathbb{R}_z^2} e^{ir\phi(x, t; z, \eta) - ir\eta \cdot y} a(x, t; z, \eta, r) r^2 dr d\eta dz, \quad (7)$$

As was stated earlier, these operators are smooth in (x, y) , and we can compute their traces by integrating these kernels over the diagonal $y = x$. Thus the (distribution) trace of $V(t)$ is a sum of terms of the form

$$Tr(\phi, a) = \int_{D \times \mathbb{R}_+ \times S^1 \times \mathbb{R}_z^2} e^{ir\phi(x, t; z, \omega) - ir\eta \cdot x} a(x, t; z, \eta, r) r^2 dr d\eta dz dx. \quad (8)$$

We want to compute the singularity in t of this trace at $t = L = 3R\sqrt{3}$, and we only need to consider t in $|t - L| < \delta$, where δ is small enough that $\{t : |t - L| < \delta\}$ contains no other lengths of periodic rays in the disk $|x| < R$.

§4. Calculation of the singularity at $t = L = 3\sqrt{3}R$

For $\eta = (\eta_1, \eta_2)$ with $|\eta| = 1$ define $\eta^\perp = (\eta_2, -\eta_1)$, the “right hand” normal. To compute the singularity at $t = L$ we only need the parametrix restricted to $R/2 - \epsilon < |z \cdot \eta^\perp| < R/2 + \epsilon$ for any fixed positive ϵ . Since the broken ray $x(t, z, \eta)$ is initially of the form $x = z + t\eta$, $\eta^\perp \cdot z > 0$ corresponds to rays going counterclockwise around $z = 0$, and $\eta^\perp \cdot z < 0$ corresponds to rays going clockwise around $z = 0$.

In the preceding section we concluded that the singularity in the wave trace at $t = L$ could be calculated from a sum of integrals of the form

$$\frac{1}{2} \sum_{\pm} \int_0^\infty r^2 dr \int_{S^1} d\eta \left(\int a_0(x, \pm t, z, \eta) e^{ir(\phi(x, \pm t, z, \eta) - x \cdot \eta)} dx dz \right). \quad (9)$$

The integral in r is to be taken in distribution sense. Until the end of this section we will consider (9) in the case that the phase ϕ is the beam phase resulting from reflecting the bicharacteristic with initial data $(x, \xi) = (z, \eta)$ three times in $|x| = R$. The amplitudes $a_0(x, t, z, \eta)$ are determined by the transport equation (4). The contributions to the singularity from the $+$ and $-$ terms in (9) are complex conjugates of each other, and from here on we only consider the “ $+$ ” term.

We assume that a_0 vanishes when $|z \cdot \eta^\perp|$ is not close to $R/2$. Note that we can assume that $\phi(x, t, z, \eta)$ is defined for all (x, z, t) when $|z \cdot \eta^\perp|$ is sufficiently close to $R/2$.

The main step in isolating the singularity is an application of the method of stationary phase to (9). For that we introduce the change of coordinates

$$x = u + v\eta + w\eta^\perp, \quad z = v\eta + w\eta^\perp, \quad u \in \mathbb{R}^2, \quad v, w \in \mathbb{R}.$$

Our objective is the elimination of the integral in (u, w) by stationary phase. To see when the phase is real and stationary in these variables note that

- i) the phase is real only when $x = x(t, z, \eta)$,
- ii) the derivative of the phase with respect to u at $x = x(t, z, \eta)$ is

$$\phi_x - \eta = \xi(t, z, \eta) - \eta,$$

which vanishes precisely when three reflections have made ξ return to its initial value. That implies $|z \cdot \eta^\perp| = R/2$. Since the reflected ray will return to z when $t = L$ and it is propagating in the direction η , $x(t, z, \eta) = z + (t - L)\eta$. Hence $u = (t - L)\eta$ and $|w| = R/2$ on the stationary set in u .

The derivative of the phase with respect to w at $x = x(t, z, \eta)$ is

$$\eta^\perp \cdot \phi_x + \eta^\perp \cdot \phi_z - \eta \cdot \eta^\perp$$

which vanishes, since $\phi_z(x(t, z, \eta), t, z, \eta) = \phi_z(x(0, z, \eta), 0, z, \eta) = \partial_z(x \cdot \eta + i|x - z|^2/2)|_{x=z} = 0$. Thus we will need to do the stationary phase computation at $(u, w) = ((t - L)\eta, \pm R/2)$.

Calculation of asymptotics by stationary phase requires the computation of the determinant of the Hessian of the phase, and here this computation is rather long. We have found it useful to consider the phase and the bicharacteristics defined for all $\eta \neq 0$ by homogeneity. That makes the Jacobian matrix

$$F(t) = \begin{pmatrix} \frac{\partial x}{\partial z}(t, z, \eta) & \frac{\partial x}{\partial \eta}(t, z, \eta) \\ \frac{\partial \xi}{\partial z}(t, z, \eta) & \frac{\partial \xi}{\partial \eta}(t, z, \eta) \end{pmatrix} =_{def} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

symplectic. Using $\phi_x(x(t, z, \eta), t, z, \eta) = \xi(t, z, \eta)$ and $\phi_z(x(t, z, \eta), t, z, \eta) = 0$ and setting $M = \phi_{xx}(x(t, z, \eta), t, z, \eta)$, one computes directly that at $x = x(t, z, \eta)$

$$H =_{def} \begin{pmatrix} \phi_{xx} & \phi_{xz} \\ \phi_{zx} & \phi_{zz} \end{pmatrix} = \begin{pmatrix} M & c - Ma \\ c^t - a^t M & a^t Ma - a^t c \end{pmatrix}.$$

Letting O_η be the matrix with columns η and η^\perp , one sees that the Hessian of the phase in (9) with respect to the variables (u, v, w) is $B^t H B$ where

$$B = \begin{pmatrix} I & O_\eta \\ 0 & O_\eta \end{pmatrix}.$$

However, we need the Hessian with respect to (u, w) . We will see that $\begin{pmatrix} \eta \\ \eta \end{pmatrix}$ is a

null vector for H , and we have $B \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \eta \\ \eta \end{pmatrix}$. Moreover, letting P_η denote the orthogonal projection of \mathbb{R}^2 onto $\langle \eta \rangle$, one computes

$$B^t \begin{pmatrix} 0 & 0 \\ 0 & P_\eta \end{pmatrix} B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$\det \begin{pmatrix} \phi_{u_1 u_1} & \phi_{u_1 u_2} & 0 & \phi_{u_1 w} \\ \phi_{u_2 u_1} & \phi_{u_2 u_2} & 0 & \phi_{u_2 w} \\ 0 & 0 & 1 & 0 \\ \phi_{w u_1} & \phi_{w u_2} & 0 & \phi_{w w} \end{pmatrix} = \det \begin{pmatrix} M & c - Ma \\ c^t - a^t M & a^t Ma - a^t c + P_\eta \end{pmatrix}. \quad (10)$$

To proceed with this computation we need to know $F(t)$. The computation begins with the formulas for $x(t, z, \eta)$ and $\xi(t, z, \eta)$ after three reflections:

$$x(t, z, \eta) = w \frac{\xi^\perp}{|\xi|} + (t + \frac{z \cdot \eta}{|\eta|} - 6\sqrt{R^2 - w^2}) \frac{\xi}{|\xi|}$$

and, setting $\eta = |\eta|(\cos \theta, \sin \theta)$,

$$\xi(t, z, \eta) = |\eta|(\cos(\theta + \pi - 6 \sin^{-1} \frac{w}{R}), \sin(\theta + \pi - 6 \sin^{-1} \frac{w}{R})).$$

One checks that $\partial_z w = \frac{\eta^\perp}{|\eta|}$ and $\partial_\eta w = -(z \cdot \eta) \frac{\eta^\perp}{|\eta|^3}$, and this implies that the Jacobian $\frac{\partial \xi}{\partial z}$ at $w = \pm R/2$ is $\frac{4\sqrt{3}}{R} |\eta| P_{\eta^\perp}$. So $c = \frac{4\sqrt{3}}{R} |\eta| P_{\eta^\perp}$. Using $\partial_\eta \theta = -\eta^\perp / |\eta|^2$, one finds that at $w = \pm R/2$

$$\frac{\partial \xi}{\partial \eta} = P_\eta + P_\eta^\perp - \frac{4\sqrt{3}}{R} \frac{z \cdot \eta}{\eta} P_{\eta^\perp} = I - \frac{4\sqrt{3}}{R} \frac{z \cdot \eta}{|\eta|} P_{\eta^\perp}$$

So $d = I - \frac{4\sqrt{3}}{R} v P_{\eta^\perp}$.

The computations of the derivatives of $x(t, z, \eta)$ are longer, but they are simplified by the observation that $|\xi(t, z, \eta)| = |\eta|$. At $w = \pm R/2$ one has

$$\begin{aligned} \frac{\partial x}{\partial z} &= P_{\eta^\perp} \mp 2\sqrt{3} \frac{\eta}{|\eta|} \langle \frac{\eta^\perp}{|\eta|}, \cdot \rangle + \frac{\eta}{|\eta|} \langle \frac{\eta}{|\eta|} \pm 2\sqrt{3} \frac{\eta^\perp}{|\eta|}, \cdot \rangle + (t - L + \frac{z \cdot \eta}{|\eta|}) \frac{4\sqrt{3}}{R} P_{\eta^\perp} \\ &= I + (t - L + \frac{z \cdot \eta}{|\eta|}) \frac{4\sqrt{3}}{R} P_{\eta^\perp}. \end{aligned}$$

So $a = I + (t - L + v) \frac{4\sqrt{3}}{R} P_{\eta^\perp}$.

To compute $\frac{\partial x}{\partial \eta}$ at $w = \pm R/2$ one uses

$$(\frac{\xi}{|\xi|})_\eta = \frac{1}{|\eta|} (1 - \frac{4\sqrt{3}}{R} \frac{z \cdot \eta}{|\eta|}) P_{\eta^\perp}$$

at $w = \pm R/2$, and the less obvious result that

$$(\frac{\xi^\perp}{|\xi|})_\eta = (-1 + \frac{4\sqrt{3}}{R} \frac{z \cdot \eta}{|\eta|}) \frac{\eta}{|\eta|^2} \langle \frac{\eta^\perp}{|\eta|}, \cdot \rangle.$$

Combining those with $\partial_\eta v = (z \cdot \eta^\perp) \frac{\eta^\perp}{|\eta|^3} = \pm \frac{R}{2|\eta|^2} \eta^\perp$, one has

$$\begin{aligned}
\frac{\partial x}{\partial \eta} &= \frac{\eta^\perp}{|\eta|} \langle -(z \cdot \eta) \frac{\eta^\perp}{|\eta|^3}, \cdot \rangle \pm \frac{R}{2} \left(-1 + \frac{4\sqrt{3}}{R} \frac{z \cdot \eta}{|\eta|} \right) \frac{\eta}{|\eta|^2} \langle \frac{\eta^\perp}{|\eta|}, \cdot \rangle \\
&+ \frac{\eta}{|\eta|} \langle (\pm \frac{R}{2|\eta|^2} \eta^\perp \mp 2\sqrt{3}(z \cdot \eta) \frac{\eta^\perp}{|\eta|^3}, \cdot \rangle + (t - L + \frac{z \cdot \eta}{|\eta|}) (1 - \frac{4\sqrt{3}}{R} \frac{z \cdot \eta}{|\eta|}) P_{\eta^\perp} \rangle |\eta|^{-1} \\
&= \frac{(t - L + \frac{z \cdot \eta}{|\eta|})}{|\eta|} (1 - \frac{4\sqrt{3}}{R} \frac{z \cdot \eta}{|\eta|}) P_{\eta^\perp} - \frac{(z \cdot \eta)}{|\eta|^2} P_{\eta^\perp}.
\end{aligned}$$

Thus

$$F(t) = \begin{pmatrix} I + (t - L + v) \frac{4\sqrt{3}}{R} P_{\eta^\perp} & \frac{(t-L)}{|\eta|} (1 - \frac{4\sqrt{3}}{R} v) P_{\eta^\perp} - \frac{4\sqrt{3}}{R} \frac{v^2}{|\eta|} P_{\eta^\perp} \\ \frac{4\sqrt{3}}{R} |\eta| P_{\eta^\perp} & I - \frac{4\sqrt{3}}{R} v P_{\eta^\perp} \end{pmatrix}. \quad (11)$$

Now we can resume the computation of the Hessian. First we compute the determinant of the Hessian. For this the only facts that we need from the computation of the symplectic matrix $F(t)$ – it is a good check on the computation to verify that it *is* symplectic – are that a , b , c and d commute with P_η with $aP_\eta = dP_\eta = P_\eta$ and $bP_\eta = cP_\eta = 0$. We will also eventually use the exact form of c . Note that since $F(t)$ is symplectic $a^t c$ and $d^t b$ are symmetric and $a^t d - c^t b = I$.

Returning to (10) we have

$$\begin{aligned}
&\begin{pmatrix} M & c - Ma \\ c^t - a^t M & a^t Ma - a^t c + P_\eta \end{pmatrix} \begin{pmatrix} I & a \\ 0 & I \end{pmatrix} = \begin{pmatrix} M & c \\ c^t - a^t M & P_\eta \end{pmatrix} \text{ and} \\
&\begin{pmatrix} I & 0 \\ a^t & I \end{pmatrix} \begin{pmatrix} M & c \\ c^t - a^t M & P_\eta \end{pmatrix} = \begin{pmatrix} M & c \\ c^t & a^t c + P_\eta \end{pmatrix}.
\end{aligned}$$

Since $M = (c + id)(a + ib)^{-1}$ (cf. [CRR]),

$$\begin{aligned}
&\begin{pmatrix} M & c \\ c^t & a^t c + P_\eta \end{pmatrix} \begin{pmatrix} a + ib & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} c + id & c \\ c^t a + ic^t b & a^t c + P_\eta \end{pmatrix} \\
&\begin{pmatrix} -a^t & I \\ I & 0 \end{pmatrix} \begin{pmatrix} c + id & c \\ c^t a + ic^t b & a^t c + P_\eta \end{pmatrix} = \begin{pmatrix} i(c^t b - a^t d) & P_\eta \\ c + id & c \end{pmatrix} = \begin{pmatrix} -iI & P_\eta \\ c + id & c \end{pmatrix}.
\end{aligned}$$

Finally

$$\begin{pmatrix} -ic + d & I \\ I & 0 \end{pmatrix} \begin{pmatrix} -iI & P_\eta \\ c + id & c \end{pmatrix} = \begin{pmatrix} 0 & P_\eta + c \\ -iI & P_\eta \end{pmatrix}.$$

From the preceding, using the exact form of c , one can read off the determinant of the Hessian of the phase (at $u = (t - L)\eta$, $w = \pm R/2$). It is

$$(-1) \left(\frac{4\sqrt{3}}{R} \right) \det((a + ib)^{-1}). \quad (12)$$

We do not need to know the Hessian matrix for the phase, but we will need to know the null space of its imaginary part. Note that $H \begin{pmatrix} a \\ I \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix}$ (note that $a^t c = c^t a$). This says that the columns of $\begin{pmatrix} a \\ I \end{pmatrix}$ are in the null space of $\text{Im}\{H\}$. Since $\text{Im}\{M\}$ is positive definite, the null space of $\text{Im}\{H\}$ can be at most two dimensional. Since the columns of $\begin{pmatrix} a \\ I \end{pmatrix}$ are linearly independent, the nullspace of $\text{Im}\{H\}$ must be the span of those columns. Hence from (11) one sees that the nullspace of $\text{Im}\{H\}$ has the orthogonal basis

$$\{(\eta, \eta), ((1 + (t - L + v)\frac{4\sqrt{3}}{R})\eta^\perp, \eta^\perp)\}. \quad (13)$$

At this point it is convenient to calculate the amplitude a_0 . Note that $\phi_t(a_0)_t - \phi_x \cdot (a_0)_x = -\frac{d}{dt}a_0(x(t, z, \eta), t, z, \eta) =_{def} -\dot{a}_0$. Hence (4) implies that, after three reflections, evaluating a_0 when $z = v\eta \pm (R/2)\eta^\perp$ gives

$$a_0(x(t, z, \eta), t, z, \eta) = (-1)^3 e^{i \int_0^t A(x(s)) \dot{x}(s) ds} e^{(\int_0^t [\phi_{tt} - \Delta\phi](x(s), s) ds)/2}. \quad (14)$$

Note that $|\phi_x| + \phi_t$ vanishes to second order on $(x, t, z, \eta) = (x(t, z, \eta), t, z, \eta)$, and on that ray $\phi_{tt} + \phi_{tx} \cdot \dot{x} = 0$ and $\phi_x = \eta/|\eta|$. Hence, differentiating $|\phi_x| + \phi_t = 0$ with respect to x and using $\phi_{tt} = -\phi_{tx} \cdot \eta/|\eta|$, we have

$$\phi_{tt} - \Delta\phi = \eta/|\eta| \cdot M\eta/|\eta| - \text{trace}(M).$$

Differentiating $\dot{x} = \xi/|\xi|$ with respect to z and η one sees that $\dot{a} + i\dot{b} = (I - P_\eta)(c + id)$. Hence, using $M = (c + id)(a + ib)^{-1}$

$$\begin{aligned} \frac{d}{dt}(\log \det(a + ib)) &= \text{trace}((\dot{a} + i\dot{b})(a + ib)^{-1}) = \text{trace}(M - P_\eta M) \\ &= \text{trace}(M - P_\eta M P_\eta) = \Delta\phi - \phi_{tt}. \end{aligned}$$

Thus, we can conclude that after three reflections

$$a_0(x(t, z, \eta), t, z, \eta) = (-1)^3 (\det(a + ib))^{-1/2} e^{i \int_0^t A(x(s)) \dot{x}(s) ds}, \quad (15)$$

where the square root is chosen continuously in t starting at 1 when $t = 0$. We have $\int_0^L A(x(s)) \dot{x}(s) ds = \alpha_\gamma$, where γ is the equilateral triangle traced by $x(s, z, \eta)$ with $z = v\eta + (R/2)\eta^\perp$ or $z = v\eta - (R/2)\eta^\perp$. Since the magnetic field vanishes in Ω , α_γ is independent of v and η , and its value when $z = v\eta + (R/2)\eta^\perp$ is the negative of its value when $z = v\eta - (R/2)\eta^\perp$.

Now we can evaluate the integral in (u, w) asymptotically by the method of stationary phase. From here on we will always have $\eta = (\cos \theta, \sin \theta)$. The standard form of the stationary phase lemma, ([Hör], Theorem 7.7.5), gives the following: if $f(y)$ is a smooth function such that $\text{Im}\{f\} \geq 0$, $f_y(y_0) = 0$ and the Hessian $f_{yy}(y_0)$ is nonsingular, then for a smooth with support in a sufficiently small neighborhood of y_0 , one has the asymptotic expansion

$$\int_{\mathbb{R}^n} e^{irf(y)} a(y) dy = \left(\frac{2\pi}{r}\right)^{n/2} \sum_{j=0}^{\infty} c_j r^{-j},$$

and the leading coefficient is given by

$$c_0 = e^{irf(y_0)} a(y_0) (\det(-if_{yy}(y_0))_*^{-1/2}. \quad (16)$$

The subscript $*$ indicates that the square root has to be chosen according to a rule determined by the eigenvalues of f_{yy} , see [Hör].

In our case we will use stationary phase to eliminate the integrations in u and w in (9) – recall that $z = v\eta + w\eta^\perp$ and $x = u + v\eta + w\eta^\perp$. The stationary point y_0 in (13) is either $(u, w) = ((t - L)\eta, R/2)$ or $(u, w) = ((t - L)\eta, -R/2)$. Since

$$\phi(x(t, z, \eta), t, z, \eta) = \phi(x(0, z, \eta), 0, z, \eta) = z \cdot \eta,$$

and we have

$$f(y_0) = \phi(x(t, z, \eta), t, z, \eta) - x(t, z, \eta) \cdot \eta$$

evaluated at $(u, w) = ((t - L)\eta, R/2)$ or $(u, w) = ((t - L)\eta, -R/2)$, it follows that $f(y_0) = -(t - L)$. The domain of integration in (u, v, w, η) is

$$\{(u, v, w, \eta) : |\eta| = 1, |u + v\eta + w\eta^\perp| \leq R \text{ and } \sqrt{w^2 + v^2} < R + \delta\}. \quad (17)$$

We consider (9) as an iterated integral with the integrations in (u, w) done first. After we use the stationary phase lemma in those integrations, the resulting integrand is evaluated at $(u, w) = ((t - L)\eta, \pm R/2)$, and, since we can assume that $|t - L|$ is smaller than δ , the domain of integration in (v, η) becomes

$$D =_{def} [-\frac{\sqrt{3}}{2}R - (t - L), \frac{\sqrt{3}}{2}R - (t - L)] \times S^1.$$

There is a problem when v is at the endpoints of its range: then the restriction $|u + v\eta + w\eta^\perp| \leq R$ does not allow integration in (u, w) over a neighborhood of the stationary points, and the stationary phase lemma does not apply. We will handle this by first extending the domain of integration in (u, w) to a neighborhood of those boundary points, and then showing that the integration over the added portion of the domain does not contribute to the leading term in the singularity. Assuming that the domain of integration has been enlarged, (12), (15) and (16) give uniformly for $(v, \eta) \in D$

$$\int_{D(v, \eta)} a_0(x, t, z, \eta) e^{ir(\phi(x, t, z, \eta) - x \cdot \eta)} dudw = \pm \frac{c(R)}{r^{3/2}} K(t) e^{-ir(t-L)} + O(\frac{1}{r^{5/2}}), \quad (18)$$

where $D(v, \eta) = \{(u, w) : |u + v\eta + w\eta^\perp| \leq R + \epsilon\}$ for ϵ small, and $c(R) = (2\pi)^{3/2} (\frac{R}{4\sqrt{3}})^{1/2} e^{-i\pi/4}$. The choice of sign \pm is determined by (15) and (16): it is -1 when the square roots of $\det(a + ib)$ implicit in (15) and (16) agree and $+1$ when they do not. The factor

$$K(t) = \exp(i \int_0^t A(x^+(s)) \cdot \dot{x}^+(s) ds) + \exp(i \int_0^t A(x^-(s)) \cdot \dot{x}^-(s) ds)$$

arises from adding the contributions from stationary points with $w = -R/2$ and $w = R/2$. The path $x^-(s)$ with $w = -R/2$ goes clockwise around the origin, and the path $x^+(s)$ with $w = R/2$ is counterclockwise. Hence $K(L) = 2 \cos(\int_\gamma A(x) \cdot dx)$, and integration over (v, η, r) gives the leading singularity in the trace at $t = L$ announced in earlier, i.e.

$$\pm \pi^3 R (2R)^{1/2} 3^{1/4} \cos\left(\int_\gamma A(x) \cdot dx\right) (t - L - i0)^{-3/2}. \quad (19)$$

This uses the distribution calculation

$$\int_0^\infty e^{-i(t-L)r} r^{1/2} dr = \frac{e^{-3\pi i/4} \Gamma(3/2)}{(t - L - i0)^{3/2}}.$$

The final step in this argument is showing that (19) really is the leading term in the singularity. Hence we need to look at the integral integral (in (u, v, w)) over the region added to derive (18). This region is given by

$$E =_{def} \{(u, v, w) : R \leq |u + v\eta + w\eta^\perp| \leq R + \epsilon, v \in [-\frac{\sqrt{3}}{2}R - (t-L), \frac{\sqrt{3}}{2}R - (t-L)]\}.$$

Note that, since the stationary points are given by $(u, w) = (t-L)\eta, \pm R/2$, the stationary points are in E only when v is at one of the endpoints of its range. Without loss of generality we assume that v is the endpoint $\frac{\sqrt{3}}{2}R - (t-L)$. We will only work on a small neighborhood of that point, since the domain only needed to be enlarged when v was in a neighborhood of that point. Likewise we will ignore all contributions beyond the second order Taylor expansion of the phase, and use the tangent plane as the boundary, since higher order corrections contribute lower order terms in r to the integrals.

To estimate the integral in (u, w) we introduce an orthogonal frame, $\{e_1, e_2, e_3\}$ with e_1 an inner (with respect to E) normal to $|x| = R$ at

$$\begin{aligned} (x, z) &= (x_0, z_0) = (u_0 + v_0\eta + w_0\eta^\perp, v_0\eta + w_0\eta^\perp) \\ &= \left(\frac{\sqrt{3}R}{2}\eta + \frac{R}{2}\eta^\perp, \left(\frac{\sqrt{3}R}{2} - (t-L)\right)\eta + \frac{R}{2}\eta^\perp\right). \end{aligned}$$

A convenient choice for this frame is

$$\begin{aligned} e_1 &= \sqrt{3}(\eta, \eta) + (\eta^\perp, \eta^\perp), \quad e_2 = \sqrt{3}(\eta, -\eta) + (\eta^\perp, -\eta^\perp) \text{ and} \\ e_3 &= (\eta, -\eta) - \sqrt{3}(\eta^\perp, -\eta^\perp). \end{aligned}$$

Note that e_2 and e_3 are orthogonal to (η, η) which spans the null space of the Hessian of the phase. Using this frame we can introduce coordinates on \mathbb{R}^4 with origin at (x_0, z_0) by setting

$$(x - x_0, z - z_0) = \tilde{u}_1 e_1 + \tilde{u}_2 e_2 + (v - v_0)(\eta, \eta) + \tilde{w} e_3.$$

Since e_2 and e_3 are orthogonal to (η, η) , it follows from (12) that the Hessian of the phase at (x_0, z_0) with respect to \tilde{u}_2 and \tilde{w} is nonsingular. Since those variables are tangential to the boundary $|x| = R$ at (x_0, z_0) – remember that we are replacing boundaries by tangent planes – the integral with respect to those variables is order r^{-1} by stationary phase. Moreover, after this use of stationary phase the phase function is evaluated at $(x, z) = (x_0, z_0) + \tilde{u}_1 e_1 + (v - v_0)(\eta, \eta)$. Hence, (see (13)) the imaginary part of the phase modulo terms which are $O(\tilde{u}_1^3)$ is given by $\alpha \tilde{u}_1^2$ with $\alpha > 0$. Now we are left with integration in (v, \tilde{u}_1) . The domain of integration in these variables is determined by the constraint $|x| \geq R$ on E . Letting $s = v_0 - v$ (note that $s \geq 0$) this constraint translates to

$$(\sqrt{3}\tilde{u}_1 - s + \frac{\sqrt{3}}{2}R)^2 + (\tilde{u}_1 + \frac{R}{2})^2 \geq R^2,$$

and it follows that for (\tilde{u}_1, s) sufficiently small we must have $\tilde{u}_1 \geq as$ for some $a > 0$ (any $a < \sqrt{3}/4$ will work). Thus the leading integral in (r, \tilde{u}_1, s) has the form

$$I(t) = \int_0^\infty \int_{as}^\infty \int_0^\infty r e^{-ir(t-L) + i(\alpha_1 + i\alpha_2)d^2(\tilde{u}_1, v)} \varphi(\tilde{u}_1, v) dr d\tilde{u}_1 ds,$$

where $d^2 \geq C\tilde{u}_1^2$, $\varphi(\tilde{u}_1, v)$ has a compact support $\varphi(\tilde{u}_1, v) = 1$ near $(0, 0)$, $\alpha_2 > 0$.

Integrating in r we get

$$I(t) = \int_0^\infty \int_{as}^\infty \frac{-1}{[-(t-L) + (\alpha_1 + i\alpha_2)d^2]^2} \varphi(\tilde{u}_1, v) d\tilde{u}_1 ds.$$

Note that $|(t-L) + (\alpha_1 + i\alpha_2)d^2|^2 \geq C_1(|t-L| + u_1^2)^2 \geq C_1|t-L|^{1+\epsilon}|u_1^2|^{1-\epsilon}$ for $0 < \epsilon < 1/2$. Then $2 - 2\epsilon > 1$ and

$$\int_{as}^\infty \frac{d\tilde{u}_1}{\tilde{u}_1^{2-2\epsilon}} = \frac{1}{(1-2\epsilon)(as)^{1-2\epsilon}}.$$

Therefore estimating the integrals in \tilde{u}_1 and s we get

$$|I(t)| \leq \frac{c}{|t-L|^{1+\epsilon}}, \quad 0 < \epsilon < \frac{1}{2}.$$

Thus $I(t)$ has a lower order singularity near $t = L$.

One should check that the contribution from the reflected beam near x_0 is also of lower order. Recall that the reflected phase ϕ^r described at the beginning of §3 equals ϕ on the boundary. Hence its Hessian with respect to the variables (\tilde{u}_2, \tilde{w}) at (x_0, z_0) is identical to the corresponding Hessian of ϕ . Moreover, the reflection rule implies that the tangential component of ϕ_x^r is equal to the tangential component of ϕ_x on the boundary. This implies that the phase $\phi^r - \eta \cdot x$ is stationary at boundary points when $\phi - \eta \cdot x$ is stationary, and we can use the stationary phase argument in the preceding paragraph again to eliminate the integrals in (\tilde{u}_2, \tilde{w}) .

This leaves an integrand for the integration in \tilde{u}_1 which is order r^{-1} as before. However, since \tilde{u}_1 is normal to the boundary the resulting phase is not stationary in \tilde{u}_1 and integration by parts in \tilde{u}_1 makes the contribution from the reflected beam of lower order. This completes the proof that (19) is the leading singularity in the wave trace at $t = L$.

§5. The Aharonov-Bohm Effect on a Torus.

The Aharonov-Bohm effect only arises when the underlying domain is not simply connected. In the previous sections the domain was an annulus. Here we consider the Schrödinger operator on a torus. Let $L = \{m_1 e_1 + m_2 e_2 : m \in \mathbb{Z}^2\}$, where $\{e_1, e_2\}$ is a basis for \mathbb{R}^2 . We assume that the lattice L has the property: For $d, d' \in L$, if $|d'| = |d|$, then $d' = \pm d$. This is a generic condition that implies that the group of isometries of L consists of lattice translations and the inversion $d \rightarrow -d$. Associated to L one has the dual lattice $L^* = \{\delta \in \mathbb{R}^2 : \delta \cdot d \in \mathbb{Z} \text{ for all } d \in L\}$.

We consider the Schrödinger operator from (1),

$$H_{A,V} = \frac{1}{2}(-i\partial_{x_1} + A_1(x))^2 + \frac{1}{2}(-i\partial_{x_2} + A_2(x))^2 - V(x),$$

acting on functions on $\mathbb{T}^2 = \mathbb{R}^2/L$. The functions $A = (A_1, A_2)$ and V are assumed to be smooth on \mathbb{T}^2 and hence they have smooth extensions to \mathbb{R}^2 satisfying $A(x+d) = A(x)$ and $V(x+d) = V(x)$ for all $d \in L$. As before we assume that the magnetic field vanishes

$$\partial_{x_2} A_1 - \partial_{x_1} A_2 = 0 \text{ on } \mathbb{T}^2. \quad (20)$$

Thus for any closed curve γ on \mathbb{T}^2 the flux

$$\alpha_\gamma = \int_\gamma A(x) \cdot dx,$$

is determined by the homology class of γ . We let γ_1 and γ_2 be a basis for the homology group, for instance

$$\gamma_j = \{te_j, t \in [0, 1)\}, \quad j = 1, 2, \quad (21)$$

and denote the corresponding fluxes by α_1 and α_2 .

Let $g(x) \in C^\infty(\mathbb{T}^2)$ be such that $|g(x)| = 1$. The conjugation of $H_{A,V}$ by the unitary operator of multiplication by $g(x)$ transforms $H_{A,V}$ to $H_{\tilde{A},V}$, where $\tilde{A} = A + ig^{-1}\nabla g$. The condition $|g(x)| = 1$ on \mathbb{T}^2 implies that $g(x) = \exp(2\pi i \delta \cdot x + \varphi(x))$, where $\delta \in L^*$ and $\varphi(x)$ is periodic. Hence $\alpha_1(\tilde{A}) = \alpha_1(A) - 2\pi \delta \cdot e_1$, $\alpha_2(\tilde{A}) = \alpha_2(A) - 2\pi \delta \cdot e_2$. Therefore if A and \tilde{A} are gauge equivalent we have

$$\alpha_j(\tilde{A}) = \alpha_j(A) \text{ modulo } 2\pi, \quad j = 1, 2. \quad (22)$$

Expanding $A(x)$ in a Fourier series we have

$$A(x) = A_0 + \sum_{\delta \in L^* \setminus \{0\}} A_\delta e^{2\pi i \delta \cdot x},$$

where $A_0 = |\mathbb{T}^2|^{-1} \int_{\mathbb{T}^2} A(x) dx$, $|\mathbb{T}^2|$ denotes the area of $\{se_1 + te_2; 0 \leq s, t \leq 1\}$. Since $\partial_{x_2} A_1 = \partial_{x_1} A_1$ we have $A(x) = A_0 + \nabla \varphi(x)$, where

$$\varphi(x) = \sum_{\delta \in L^* \setminus \{0\}} \frac{\delta \cdot A_\delta}{2\pi i \delta \cdot \delta} e^{2\pi i \delta \cdot x}.$$

Therefore when (20) holds $A(x)$ is gauge equivalent to the constant potential A_0 . Two constant magnetic potentials A_0 and \tilde{A}_0 are not gauge equivalent if (22) does not hold. When \tilde{A}_0 is not gauge equivalent to either A_0 or $-A_0$ the potentials A_0 and \tilde{A}_0 have a different physical impact, in particular, the spectra of $H_{A_0, V}$ and $H_{\tilde{A}_0, V}$ are not the same.

The last assertion is a consequence of the following theorem.

Theorem 5.1. Suppose (20) holds. The spectrum of $H_{A, V}$ as a self-adjoint operator on $L^2(\mathbb{T}^2)$ determines $\cos \alpha_1$ and $\cos \alpha_2$, where $\alpha_j = \int_{\gamma_j} A(x) \cdot dx$, $j = 1, 2$.

Theorem 5.1 complements the results of [G], [ER1] and [E1]. In particular it shows that, if A and \tilde{A} give rise to zero magnetic fields on \mathbb{T}^2 but different values for $\cos \alpha_1$ and $\cos \alpha_2$, the Schrödinger operators, $H_{A, V}$ and $H_{\tilde{A}, V}$ will have different spectra. This proves the Aharonov-Bohm effect on the torus.

Proof of Theorem 5.1. As in the preceding sections we start with the wave trace formula

$$\sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j}) = \int_{\mathbb{T}^2} E_{\mathbb{T}^2}(x, x, t) dx,$$

where $\{\lambda_j\}_{j=1}^{\infty}$ is the spectrum of $H_{A, V}$ on \mathbb{T}^2 and $E_{\mathbb{T}^2}(x, y, t)$ is the solution to $E_{tt} + H_{A, V} E = 0$ on $\mathbb{T}^2 \times \mathbb{R}$ satisfying $E(x, y, 0) = \delta(x - y)$ and $E_t(x, y, 0) = 0$. Note that

$$E_{\mathbb{T}^2}(x, y, t) = \sum_{d \in L} E_{\mathbb{R}^2}(x + d, y, t),$$

where $E_{\mathbb{R}^2}$ is the solution to $E_{tt} + H_{A, V} E = 0$ on $\mathbb{R}^2 \times \mathbb{R}$ satisfying $E(x, y, 0) = \delta(x - y)$ and $E_t(x, y, 0) = 0$ when $H_{A, V}$ has been extended to \mathbb{R}^2 by making its coefficients periodic, i.e. $A(x + d) = A(x)$ and $V(x + d) = V(x)$ for all $d \in L$. Hence

$$\int_{\mathbb{T}^2} E_{\mathbb{T}^2}(x, x, t) dx = \sum_{d \in L} \int_{\mathbb{T}^2} E_{\mathbb{R}^2}(x + d, x, t) dx. \quad (23)$$

Since $E_{\mathbb{R}^2}$ is smooth off the cone $|x - y|^2 = t^2$, and our assumption on L implies that only two lattice vectors can have $|d|^2 = t^2$ for a fixed value of t , the singularity in the wave trace at $t = |d|$, must come from (cf. [ERT], [ER2])

$$\int_{\mathbb{T}^2} E_{\mathbb{R}^2}(x + d, x, t) dx + \int_{\mathbb{T}^2} E_{\mathbb{R}^2}(x - d, x, t) dx.$$

To compute the leading singularities in this trace we will use the Hadamard-Hörmander parametrix (cf. [Hör]). We have

$$E_{\mathbb{R}^2}(x, y, t) = \partial_t(E_+(x, y, t) - E_+(x, y, -t)),$$

where E_+ is the forward fundamental solution. The Hadamard-Hörmander parametrix construction for E_+ writes E_+ as an asymptotic sum of terms with increasing regularity. The first term is $a_0(x, y)e_0(|x - y|, t)$, where

$$e_0 = \frac{1}{2\sqrt{\pi}}(t^2 - |x - y|^2)_+^{-1/2} \text{ when } t > 0 \text{ and } e_0 = 0 \text{ when } t < 0, \text{ and}$$

$$a_0(x, y) = \exp(i \int_0^1 (x - y) \cdot A(y + s(x - y)) ds).$$

Therefore (cf. [ER1]) the singularity of the trace at $t = |d|$ determines $I(d) + I(-d)$ where

$$I(d) = \int_{\mathbb{T}^2} \exp(i \int_0^1 d \cdot A(x + sd) ds) dx.$$

Since $A(x) = A_0 + \nabla\varphi(x)$, where $\varphi(x)$ is periodic, we have

$$\int_0^1 d \cdot A(x + sd) ds = d \cdot A_0 \quad \text{since} \quad \int_0^1 d \cdot \nabla\varphi(x + sd) ds = 0$$

Therefore $I(d) = e^{id \cdot A_0} |\mathbb{T}^2|$ and hence the singularity of the wave trace at $t = |d|$ determines $\cos(A_0 \cdot d)$ for all $d \in L$. In particular, when $d = e_j$ and $\gamma_j = \{te_j, t \in [0, 1)\}$, $j = 1, 2$, we get $\alpha_j = \int_{\gamma_j} A(x) \cdot dx = e_j \cdot A_0$. Thus the singularities of the wave trace when $t = |e_j|$ determine $\cos \alpha_j$ for $j = 1, 2$. When $V(x) = V(-x)$, then $H_{A_0, V}$ and $H_{-A_0, V}$ are isospectral and one can only recover $\cos \alpha_j$, $j = 1, 2$, from the spectrum. When V is not even, the question of whether one could recover $\exp(i\alpha_j)$, $j = 1, 2$, from the spectrum is open.

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